

# On the regularity and partial regularity of extremal solutions of a Lane-Emden system

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## Abstract

In this paper, we consider the system  $-\Delta u = \lambda(v+1)^p$ ,  $-\Delta v = \gamma(u+1)^\theta$  on a smooth bounded domain  $\Omega$  in  $\mathbb{R}^N$  with the Dirichlet boundary condition  $u = v = 0$  on  $\partial\Omega$ . Here  $\lambda, \gamma$  are positive parameters. Let  $x_0$  be the largest root of the polynomial

$$H(x) = x^4 - \frac{16p\theta(p+1)(\theta+1)}{(p\theta-1)^2}x^2 + \frac{16p\theta(p+1)(\theta+1)(p+\theta+2)}{(p\theta-1)^3}x - \frac{16p\theta(p+1)^2(\theta+1)^2}{(p\theta-1)^4}.$$

We show that the extremal solutions associated to the above system are bounded provided  $N < 2 + 2x_0$ . This improves the previous work in [8]. We also prove that, if  $N \geq 2 + 2x_0$ , then the singular set of any extremal solution has Hausdorff dimension less or equal to  $N - (2 + 2x_0)$ .

*Keywords:* Extremal solution, stable solution, regularity and partial regularity.

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## 1. Introduction

In this paper, we are interested in the regularity and partial regularity of extremal solutions to the following system:

$$\begin{cases} -\Delta u &= \lambda(v+1)^p & \text{in } \Omega \\ -\Delta v &= \gamma(u+1)^\theta & \text{in } \Omega \\ u &= v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$ ,  $\lambda, \gamma$  are positive parameters and  $p, \theta > 1$ . In particular, we examine when the extremal solutions of (1.1) are smooth. Applying standard elliptic theory, it is sufficient to show that the extremal solutions are bounded. The nonlinearities we examine naturally fit into the following general assumptions:

$$(R) \quad f \text{ is smooth, positive, increasing, convex in } [0, \infty), \text{ and } \lim_{u \rightarrow \infty} \frac{f(u)}{u} = \infty.$$

Recalling that the scalar analog of the system (1.1) is given by

$$(Q)_\lambda \quad -\Delta u = \lambda f(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

When the nonlinearity  $f$  satisfies (R), it is well known that there exists a finite positive critical parameter  $\lambda^*$  such that for all  $0 < \lambda < \lambda^*$  there exists a smooth minimal solution  $u_\lambda$  of  $(Q)_\lambda$ .

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Here the minimal solution means in the pointwise sense. In addition, the minimal solution  $u_\lambda$  is semi-stable in the sense that

$$\int_{\Omega} \lambda f'(u_\lambda) \psi^2 dx \leq \int_{\Omega} |\nabla \psi|^2 dx, \quad \forall \psi \in H_0^1(\Omega).$$

Moreover, the map  $\lambda \mapsto u_\lambda(x)$  is increasing on  $[0, \lambda^*)$ . This allows one to define  $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ , the so-called extremal solution, which is shown to be the unique weak solution of  $(Q)_{\lambda^*}$ , and there is no weak solution of  $(Q)_\lambda$  for  $\lambda > \lambda^*$ . The regularity and properties of extremal solution to  $(Q)_\lambda$  have attracted a lot of attention. It is known that it depends on the nonlinearity  $f$ , the dimension  $N$  and the geometry of the domain  $\Omega$ . See for instance [2, 3, 4, 5, 18, 19, 21, 23, 24].

The situation is much less understood for the corresponding elliptic system. Consider the generalization of (1.1) as follows:

$$(P)_{\lambda, \gamma} \quad \begin{cases} -\Delta u &= \lambda f(v) & \text{in } \Omega \\ -\Delta v &= \gamma g(u) & \text{in } \Omega \\ u &= v = 0 & \text{on } \partial\Omega. \end{cases}$$

Define  $\mathcal{Q} := \{(\lambda, \gamma) : \lambda, \gamma > 0\}$  and

$$\mathcal{U} := \{(\lambda, \gamma) \in \mathcal{Q} : \text{there exists a smooth solution } (u, v) \text{ of } (P)_{\lambda, \gamma}\}.$$

Set  $\Upsilon := \partial\mathcal{U} \cap \mathcal{Q}$ , which plays the role of the extremal parameter  $\lambda^*$ . As shown by Montenegro [20], if  $f$  and  $g$  satisfy (R), then

1.  $\mathcal{U}$  is nonempty. For all  $(\lambda, \gamma) \in \mathcal{U}$ , there is a minimal solution of  $(P)_{\lambda, \gamma}$ .
2. For each  $0 < \sigma < \infty$  there is some  $0 < \lambda_\sigma^* < \infty$  such that  $\mathcal{U} \cap \{(\lambda, \sigma\lambda) : 0 < \lambda\}$  is given by  $\{(\lambda, \sigma\lambda) : 0 < \lambda < \lambda_\sigma^*\} \cup \mathcal{H}$  where  $\mathcal{H}$  is either the empty set or  $\{(\lambda_\sigma^*, \sigma\lambda_\sigma^*)\}$ . The map  $\sigma \mapsto \lambda_\sigma^*$  is bounded on compact subsets of  $(0, \infty)$ .
3. Fix  $0 < \sigma < \infty$  and let  $(u_\lambda, v_\lambda)$  denote the minimal solution of  $(P)_{\lambda, \sigma\lambda}$  for  $0 < \lambda < \lambda_\sigma^*$ . Then  $u_\lambda, v_\lambda$  are increasing in  $\lambda$  and

$$u^*(x) := \lim_{\lambda \nearrow \lambda_\sigma^*} u_\lambda(x), \quad v^*(x) := \lim_{\lambda \nearrow \lambda_\sigma^*} v_\lambda(x) \quad (1.2)$$

is always a weak solution to  $(P)_{\lambda_\sigma^*, \sigma\lambda_\sigma^*}$ .

In addition, let  $(\lambda, \gamma) \in \mathcal{U}$ , the minimal solution  $(u, v)$  of  $(P)_{\lambda, \gamma}$  is semi-stable in the sense that there are  $0 < \zeta, \chi \in H_0^1(\Omega)$  and  $\eta \geq 0$  such that

$$-\Delta \zeta = \lambda f'(v) \chi + \eta \zeta, \quad -\Delta \chi = \gamma g'(u) \zeta + \eta \chi \quad \text{in } \Omega. \quad (1.3)$$

See [20] and also [8] for an alternative proof of (1.3). Moreover, we have the following useful inequality, see Lemma 1 in [8] and Lemma 3 in [15].

**Lemma 1.1.** *Let  $(u, v)$  denote a semi-stable solution of  $(P)_{\lambda, \gamma}$  in the sense of (1.3). Then*

$$\sqrt{\lambda\gamma} \int_{\Omega} \sqrt{f'(v)g'(u)} \phi^2 \leq \int_{\Omega} |\nabla \phi|^2, \quad \forall \phi \in H_0^1(\Omega). \quad (1.4)$$

For example, when  $f(t) = g(t) = e^t$ , it was shown in [15] that for  $1 \leq N \leq 9$ , the extremal solution  $(u^*, v^*)$  is smooth, see also [7, 13]. Furthermore, if  $N \geq 10$ , Dávila and Goubet showed that the Hausdorff dimension of the singular set of any extremal solution is less or equal to  $N - 10$ . For the polynomial system (1.1), Cowan proved in [8]:

**Theorem A.** *Suppose that  $1 < p \leq \theta$ ,  $(\lambda^*, \gamma^*) \in \Upsilon$ . Then, the extremal solution  $(u^*, v^*)$  of (1.1) is bounded provided  $N < 2 + \frac{4(\theta+1)t_0}{p\theta-1}$ , where*

$$t_0 = \sqrt{\frac{p\theta(p+1)}{\theta+1}} + \sqrt{\frac{p\theta(p+1)}{\theta+1} - \frac{p\theta(p+1)}{\theta+1}}. \quad (1.5)$$

Consequently, the extremal solutions are smooth for any  $1 < p \leq \theta$  provided  $N \leq 10$ .

A main idea in [8] is to use the stability inequality (1.4). This technique was used to consider various Liouville theorem and regularity of extremal solutions for elliptic systems and biharmonic equations, see for example [9, 11, 6, 16, 17, 15, 7, 10].

Our main concern here is to improve Cowan's result.

**Theorem 1.1.** *Let  $(\lambda^*, \gamma^*) \in \Upsilon$  and  $(u^*, v^*)$  denote the associated extremal solution of (1.1). Suppose that  $N < 2 + 2x_0$ , where  $x_0$  be the largest root of the polynomial  $H(x) =$*

$$x^4 - \frac{16p\theta(p+1)(\theta+1)}{(p\theta-1)^2}x^2 + \frac{16p\theta(p+1)(\theta+1)(p+\theta+2)}{(p\theta-1)^3}x - \frac{16p\theta(p+1)^2(\theta+1)^2}{(p\theta-1)^4}. \quad (1.6)$$

Then  $u^*, v^*$  are bounded. In particular, the extremal solutions are smooth provided  $N \leq 10$ .

Using Remark 2.1 below, we see that  $2t_0 \frac{\theta+1}{p\theta-1} \leq x_0$  for any  $1 < p \leq \theta$ , with equality if and only if  $p = \theta$ , where  $t_0$  is given by (1.5), so our result improves Theorem A.

To prove Theorem 1.1, we will use the following Souplet type pointwise estimate between  $u$  and  $v$ , solution of (1.1). See Lemma 2 in [8].

**Lemma 1.2.** *Let  $(u, v)$  denote a smooth solution of (1.1) and suppose that  $\theta \geq p > 1$ . Let*

$$\alpha := \max \left\{ 0, \left( \frac{\gamma(p+1)}{\lambda(\theta+1)} \right)^{\frac{1}{p+1}} - 1 \right\}.$$

Then

$$(v+1+\alpha)^{p+1} \geq \frac{\gamma(p+1)}{\lambda(\theta+1)}(u+1)^{\theta+1} \quad \text{in } \Omega. \quad (1.7)$$

Obviously, as  $v > 0$  and  $\alpha \geq 0$ , we have

$$(v+1)^{p+1} \geq \left( \frac{v+1+\alpha}{\alpha+1} \right)^{p+1} \geq \frac{\gamma(p+1)}{\lambda(\theta+1)(\alpha+1)^{p+1}}(u+1)^{\theta+1} \quad \text{in } \Omega. \quad (1.8)$$

In the spirit of [13], we are also interested in the partial regularity for extremal solutions. Let  $(u^*, v^*)$  be an extremal solution of (1.1), a point  $x \in \Omega$  is said regular if there exists a neighborhood of  $x$  on which  $u^*$  and  $v^*$  are bounded; Otherwise  $x$  is said singular. Denote by  $\mathcal{S}$  the set of singular points of  $(u^*, v^*)$ . By definition, the regular set  $\Omega \setminus \mathcal{S}$  is open and by elliptic regularity,  $u^*, v^*$  are smooth in  $\Omega \setminus \mathcal{S}$ .

**Theorem 1.2.** Assume that  $N \geq 2 + 2x_0$ , where  $x_0$  is that in Theorem 1.1. Let  $(u^*, v^*)$  denote an extremal solution of (1.1), i.e. with  $(\lambda^*, \gamma^*) \in \Upsilon$ , then the Hausdorff dimension of its singular set  $\mathcal{S}$  is less or equal to  $N - (2 + 2x_0)$ .

**Remark 1.1.** If  $p = 1$  or  $\theta = 1$  and  $p\theta > 1$ , following the proofs of Theorems 1.1-1.2, we can show that the results remain true. In other words, Theorems 1.1-1.2 hold true for  $p, \theta \geq 1$  verifying  $p\theta > 1$ .

This paper is organized as follows. We prove Theorem 1.1 in Section 2. The Section 3 is devoted to the proof of Theorem 1.2.

## 2. Proof of Theorem 1.1

First we remark that the polynomial  $H$  is completely symmetric in  $p$  and  $\theta$ . Hence we assume from now on  $1 < p \leq \theta$ , without loss of generality.

The following Lemma plays an important role in dealing with Theorem 1.1, where we use some ideas from [16, 17, 6]. Let  $(\lambda^*, \gamma^*) \in \Upsilon$  and  $\sigma := \frac{\gamma^*}{\lambda^*}$ . Define  $\Gamma_\sigma := \{(\lambda, \sigma\lambda) : \frac{\lambda^*}{2} < \lambda < \lambda^*\}$  and denote  $(u^*, v^*)$  the extremal solution associated to  $(\lambda^*, \gamma^*)$  for (1.1) defined by (1.2), i.e.  $(u^*, v^*)$  is the pointwise limit of the minimal solutions along the ray  $\Gamma_\sigma$  as  $\lambda \nearrow \lambda^*$ .

**Lemma 2.1.** Let  $(\lambda^*, \gamma^*) \in \Upsilon$ ,  $\sigma = \frac{\gamma^*}{\lambda^*}$ , let  $(u, v)$  denote the minimal solution of (1.1) for  $(\lambda, \gamma) \in \Gamma_\sigma$ . Define

$$L(s) := s^4 - \frac{16p\theta(p+1)}{\theta+1}s^2 + \frac{16p\theta(p+1)(p+\theta+2)}{(\theta+1)^2}s - \frac{16p\theta(p+1)^2}{(\theta+1)^2}. \quad (2.1)$$

Then for any  $s > p+1$  verifying  $L(s) < 0$ , there exists  $C_s < \infty$  such that for any  $(\lambda, \gamma) \in \Gamma_\sigma$ , there holds

$$\int_{\Omega} (u+1)^{\frac{\theta-1}{2}} (v+1)^{\frac{p+2s-1}{2}} + \int_{\Omega} (u+1)^{\theta + \frac{(\theta+1)(s-1)}{p+1}} \leq C_s. \quad (2.2)$$

**Proof.** We handle only the first integral in (2.2), since the second estimate is an immediate consequence of the first one thanks to (1.8). Inserting  $\phi := (u+1)^{\frac{q+1}{2}} - 1$  with  $q > 0$  into (1.4), we obtain

$$\sqrt{\lambda\gamma p\theta} \int_{\Omega} (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta-1}{2}} \left[ (u+1)^{\frac{q+1}{2}} - 1 \right]^2 \leq \frac{(q+1)^2}{4} \int_{\Omega} (u+1)^{q-1} |\nabla u|^2. \quad (2.3)$$

On the other hand, multiplying the first equation of (1.1) by  $(u+1)^q - 1$ , we get

$$\frac{(q+1)^2}{4} \int_{\Omega} (u+1)^{q-1} |\nabla u|^2 = \frac{(q+1)^2 \lambda}{4q} \int_{\Omega} (v+1)^p \left[ (u+1)^q - 1 \right]. \quad (2.4)$$

Combining (2.3), (2.4) and dropping some positive terms, there holds

$$\sqrt{\lambda\gamma} a_1 J_1 \leq \lambda \int_{\Omega} (v+1)^p (u+1)^q + 2\sqrt{\lambda\gamma} a_1 I_1, \quad (2.5)$$

where

$$a_1 = \frac{4q\sqrt{p\theta}}{(q+1)^2}, \quad J_1 := \int_{\Omega} (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta+2q+1}{2}}, \quad I_1 := \int_{\Omega} (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta+q}{2}}.$$

Similarly, using  $\phi := (v+1)^{\frac{r+1}{2}} - 1$  in (1.4) with  $r > 0$ , we obtain

$$\sqrt{\lambda\gamma}a_2J_2 \leq \gamma \int_{\Omega} (u+1)^{\theta}(v+1)^r + 2\sqrt{\lambda\gamma}a_2I_2 \quad (2.6)$$

where

$$a_2 = \frac{4r\sqrt{p\theta}}{(r+1)^2}, \quad J_2 := \int_{\Omega} (v+1)^{\frac{p+2r+1}{2}}(u+1)^{\frac{\theta-1}{2}}, \quad I_2 := \int_{\Omega} (v+1)^{\frac{p+r}{2}}(u+1)^{\frac{\theta-1}{2}}.$$

Fix now

$$q = \frac{(\theta+1)r}{p+1} + \frac{\theta-p}{p+1}, \quad \text{or equivalently } q+1 = \frac{(\theta+1)(r+1)}{p+1}. \quad (2.7)$$

Let  $r > p$  and so  $q > \theta$ , we claim that for any  $\epsilon > 0$ , there exists  $C_{\epsilon} > 0$  independent of  $(\lambda, \gamma) \in \Gamma_{\sigma}$  such that

$$I_1 \leq \epsilon J_1 + \epsilon \int_{\Omega} (v+1)^p(u+1)^q + C_{\epsilon}, \quad I_2 \leq \epsilon J_2 + \epsilon \int_{\Omega} (u+1)^{\theta}(v+1)^r + C_{\epsilon}. \quad (2.8)$$

Indeed, using successively Young's inequality for  $(u+1)^{\frac{\theta+q}{2}}$  and  $(v+1)^{\frac{p-1}{2}}$ , we get

$$\begin{aligned} I_1 &\leq \epsilon J_1 + C_{\epsilon} \int_{\Omega} (v+1)^{\frac{p-1}{2}} \leq \epsilon J_1 + \epsilon \int_{\Omega} (v+1)^p + C_{\epsilon} \\ &\leq \epsilon J_1 + \epsilon \int_{\Omega} (v+1)^p(u+1)^q + C_{\epsilon}. \end{aligned}$$

The estimate for  $I_2$  is similar, so we omit it. Inserting (2.8) into (2.5) and (2.6) respectively, we get (for  $\epsilon < 1/2$ )

$$J_1 \leq \frac{1}{A_1} \int_{\Omega} (v+1)^p(u+1)^q + C_{\epsilon}, \quad J_2 \leq \frac{1}{A_2} \int_{\Omega} (u+1)^{\theta}(v+1)^r + C_{\epsilon},$$

with

$$A_1 = \frac{a_1(1-2\epsilon)}{\sqrt{\frac{\lambda}{\gamma}} + 2a_1\epsilon}, \quad A_2 = \frac{a_2(1-2\epsilon)}{\sqrt{\frac{\gamma}{\lambda}} + 2a_2\epsilon}.$$

Hence

$$J_1 + A_2^{\frac{2(r+1)}{p+1}} J_2 \leq \frac{1}{A_1} \int_{\Omega} (u+1)^q(v+1)^p + A_2^{\frac{2r+1-p}{p+1}} \int_{\Omega} (u+1)^{\theta}(v+1)^r + C_{\epsilon}. \quad (2.9)$$

By (2.7),

$$q - \frac{\theta-1}{2} = q+1 - \frac{\theta+1}{2} = (q+1) \left[ 1 - \frac{p+1}{2(r+1)} \right].$$

Using Young's inequality, there holds

$$\begin{aligned} &\frac{1}{A_1} \int_{\Omega} (u+1)^q(v+1)^p \\ &= \int_{\Omega} (u+1)^{\frac{\theta-1}{2}}(v+1)^{\frac{p-1}{2}}(u+1)^{(q+1)\left(1-\frac{p+1}{2(r+1)}\right)} \frac{(v+1)^{\frac{p+1}{2}}}{A_1} \\ &\leq \int_{\Omega} (u+1)^{\frac{\theta-1}{2}}(v+1)^{\frac{p-1}{2}} \left[ \frac{2r+1-p}{2(r+1)}(u+1)^{q+1} + \frac{p+1}{2(r+1)} A_1^{-\frac{2(r+1)}{p+1}} (v+1)^{r+1} \right] \\ &= \frac{2r+1-p}{2(r+1)} J_1 + \frac{p+1}{2(r+1)} A_1^{-\frac{2(r+1)}{p+1}} J_2. \end{aligned}$$

Similarly we have

$$A_2^{\frac{2r+1-p}{p+1}} \int_{\Omega} (u+1)^{\theta} (v+1)^r \leq \frac{p+1}{2(r+1)} J_1 + \frac{2r+1-p}{2(r+1)} A_2^{\frac{2(r+1)}{p+1}} J_2.$$

Combining the above two estimates with (2.9), we derive that

$$A_2^{\frac{2(r+1)}{p+1}} J_2 \leq \left[ \frac{2r+1-p}{2(r+1)} A_2^{\frac{2(r+1)}{p+1}} + \frac{p+1}{2(r+1)} A_1^{\frac{-2(r+1)}{p+1}} \right] J_2 + C_{\epsilon},$$

hence

$$\frac{p+1}{2(r+1)} \left[ (A_1 A_2)^{\frac{2(r+1)}{p+1}} - 1 \right] J_2 \leq C_{\epsilon}.$$

Thus  $J_2 \leq C_{\epsilon}$  if  $A_1 A_2 > 1$ . Suppose that  $a_1 a_2 > 1$ , we can take  $\epsilon > 0$  sufficiently small so that  $A_1 A_2 > 1$ .

Denote  $s = r + 1$ . Using (2.7), we can check directly that  $a_1 a_2 > 1$  is equivalent to  $L(s) < 0$ . We conclude then for all  $s > p + 1$  verifying  $L(s) < 0$ , there is  $C_s > 0$  such that for any  $(\lambda, \gamma) \in \Gamma_{\sigma}$ ,

$$\int_{\Omega} (u+1)^{\frac{\theta-1}{2}} (v+1)^{\frac{p+1}{2}} (v+1)^{s-1} = J_2 \leq C_s. \quad (2.10)$$

So we are done.  $\square$

**Remark 2.1.** Let  $L$  be given by (2.1) and  $H$  be given by (1.6). A direct computation yields

$$H(x) = \left( \frac{\theta+1}{p\theta-1} \right)^4 L(s), \quad \text{if } x = \frac{\theta+1}{p\theta-1} s.$$

Denote  $s_0$  the largest root of  $L$ , then  $x_0 = \frac{\theta+1}{p\theta-1} s_0$  is the largest root of  $H$ , and  $H(x) < 0$  if and only if  $L(s) < 0$ . Moreover, there holds

$$L(2t_0) = \frac{16p\theta(p+1)(\theta-p)}{(\theta+1)^2} (1-2t_0).$$

So  $L(2t_0) < 0$  for  $p < \theta$ . As  $\lim_{s \rightarrow \infty} L(s) = \infty$ , it follows that  $2t_0 < s_0$ . If  $p = \theta$ , we have

$$L(s) = s^4 - 16p^2 s^2 + 32p^2 s - 16p^2 = (s^2 + 4ps - 4p)(s^2 - 4ps + 4p).$$

For any  $p > 1$ , we check readily that  $2t_0 = 2p + 2\sqrt{p^2 - p}$  is the largest root of  $L$ .

**Proof of Theorem 1.1 completed.** Let  $(\lambda^*, \gamma^*) \in \Upsilon$  and  $\sigma = \frac{\gamma^*}{\lambda^*}$ . Denote  $(u, v)$  the minimal solution of (1.1) with  $(\lambda, \gamma) \in \Gamma_{\sigma}$ . Applying Lemma 2.1, if  $p+1 < s < s_0$ , there exists  $C_s > 0$  such that

$$\frac{1}{\gamma} \int_{\Omega} |\nabla v|^2 = \int_{\Omega} (u+1)^{\theta} v \leq \int_{\Omega} (u+1)^{\theta} (v+1)^{s-1} \leq C_s,$$

passing to the limit, we see that  $v^* \in H_0^1(\Omega)$ . Moreover,

$$\frac{-\Delta v^*}{\gamma^*} = (u^* + 1)^{\theta} = \frac{(u^* + 1)^{\theta}}{v^* + 1} v^* + \frac{(u^* + 1)^{\theta}}{v^* + 1} \quad \text{in } \Omega.$$

By standard elliptic theory, to show the boundedness of  $v^*$ , it is sufficient to prove that  $\frac{(u^*+1)^\theta}{v^*+1} \in L^T(\Omega)$  for some  $T > \frac{N}{2}$ . Using (1.8) and passing to the limit, we see that there is some  $C > 0$  such that

$$\frac{(u^*+1)^\theta}{v^*+1} \leq C(u^*+1)^{\frac{p\theta-1}{p+1}} \quad \text{a.e. in } \Omega.$$

According to the estimate (2.2) which holds also with  $u^*$ , it follows that  $\frac{(u^*+1)^\theta}{v^*+1} \in L^T(\Omega)$  for some  $T > \frac{N}{2}$  provided

$$\frac{(p\theta-1)}{p+1} \frac{N}{2} < \theta + \frac{(\theta+1)(s_0-1)}{p+1}, \quad \text{or equivalently } N < 2 + 2x_0 \text{ where } x_0 = \frac{\theta+1}{p\theta-1}s_0,$$

This is just the desired result. Moreover, using Remark 2.1 and adopting the proof of Remark 2 in [8], we can easily show that

$$x_0 \geq 2t_0 \frac{\theta+1}{p\theta-1} > 4, \quad \forall \theta \geq p > 1.$$

This means that if  $N \leq 10$ ,  $(u^*, v^*)$  is bounded. □

### 3. Proof of Theorem 1.2

#### 3.1. Some preparations

We establish first some properties of the polynomial  $L$  defined by (2.1). Recall that without loss of generality, we can assume  $1 < p \leq \theta$ .

**Lemma 3.1.** *Let  $1 < p \leq \theta$ , then  $L(2) < 0$  and  $L$  has a unique root  $s_0$  in  $(2, \infty)$ . Moreover, we have  $p+1 < 2\theta \frac{p+1}{\theta+1} < s_0$ .*

**Proof.** As  $1 < p \leq \theta$ , we have

$$\begin{aligned} L(2) &= 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)(p+\theta+2)}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &= 16 - \frac{64p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)}{(\theta+1)} + \frac{32p\theta(p+1)^2}{(\theta+1)^2} - \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &= 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)^2}{(\theta+1)^2} \\ &\leq 16 - \frac{32p\theta(p+1)}{(\theta+1)} + \frac{16p\theta(p+1)}{(\theta+1)} \\ &= 16 \frac{(1-p^2)\theta + (1-p\theta)}{(\theta+1)} < 0. \end{aligned}$$

Very similarly, we can check that

$$L'(2) \leq 32 - \frac{32p\theta(p+1)}{(\theta+1)} < 0 \quad \text{and} \quad L''(s) = 12s^2 - \frac{32p\theta(p+1)}{\theta+1}.$$

Then  $L''$  could change at most once the sign from negative to positive in  $[2, \infty)$ , hence  $L$  admits a unique root in  $(2, \infty)$ . In addition, direct calculations yield to

$$L(p+1) = -\frac{(p+1)^2(5p\theta + \theta + p + 1)(3p\theta - \theta - p - 1)}{(\theta+1)^2} < 0,$$

and

$$-\frac{(\theta+1)^4}{16\theta(p+1)^2}L\left(2\theta\frac{p+1}{\theta+1}\right) = (3p^2-1)\theta^3 + (2p^2-p)\theta^2 - 2(p^2+p)\theta + p =: K(p, \theta).$$

It's not difficult to check that for any  $1 < p \leq \theta$ ,  $K(p, \theta) > K(p, p) > 0$ , hence  $L(2\theta\frac{p+1}{\theta+1}) < 0$ , which means  $2\theta\frac{p+1}{\theta+1} < s_0$ .  $\square$

Using similar ideas as in the proof for Lemma 2.1 and following the proof of Lemma 3.1 in [16], we can claim

**Lemma 3.2.** *Let  $(u, v)$  be a stable solution of (1.1). Then, for any  $s > \frac{p+1}{2}$  verifying  $L(s) < 0$ , there exists  $C < \infty$  such that if  $B_R(y) \subset \Omega$  for  $R > 0$ , then*

$$\int_{B_{R/2}(y)} (u+1)^\theta (v+1)^{s-1} dx \leq \frac{C}{R^2} \int_{B_R(y)} (v+1)^s dx.$$

We will need also the following well known elliptic estimate. Denote  $B_r := B_r(0)$  for any  $r > 0$ .

**Lemma 3.3.** *For  $1 \leq t < \frac{N}{N-2}$ , there exists  $C > 0$  such that for any  $w \in W^{2,1}(B_{2R})$  with  $R > 0$ , we have*

$$\left( \int_{B_R} |w|^t dx \right)^{\frac{1}{t}} \leq CR^{N(\frac{1}{t}-1)+2} \int_{B_{2R}} |\Delta w| dx + CR^{N(\frac{1}{t}-1)} \int_{B_{2R}} |w| dx.$$

As a consequence of the two above Lemmas, we state

**Lemma 3.4.** *Let  $(u, v)$  be a stable solution of (1.1). Then, for any  $2 \leq s < \frac{N}{N-2}s_0$ , there are  $\ell \in \mathbb{N}$  and  $C > 0$  such that if  $B_R(y) \subset \Omega$ , then*

$$\left( R^{-N} \int_{B_{2^{-\ell}R}(y)} (v+1)^s dx \right)^{\frac{2}{s}} \leq CR^{-N} \int_{B_R(y)} (v+1)^2 dx.$$

**Proof.** The proof of Lemma 3.4 is very similar to that for Proposition 1 in [6], (see also Lemma 3.3 in [16] for a more general setting). It follows from the application of Lemma 3.3 with  $w = (v+1)^s$ . We use also Lemma 3.2 to control the integral

$$\int_{B_{R/2}(y)} (u+1)^\theta (v+1)^{s-1} dx$$

appeared after multiplying the equation of  $v$  by  $(v+1)^{s-1}\phi^2$ , where  $\phi$  is a suitable cut off function. We omit the details here.  $\square$

**Remark 3.1.** *Let  $(\lambda^*, \gamma^*) \in \Upsilon$  and  $\sigma = \frac{\gamma^*}{\lambda^*}$ . Suppose that  $(u, v)$  is a stable solution of (1.1) with  $(\lambda, \gamma) \in \Gamma_\sigma$ . Although the constant  $C$  appearing in Lemma 3.2 as well as in Lemma 3.4 depends on  $\lambda, \gamma$ , it remains bounded as  $\lambda \nearrow \lambda^*$ .*

### 3.2. $\varepsilon$ -regularity.

Inspired by [13], we prove the following  $\varepsilon$ -regularity result which is crucial in proving Theorem 1.2. Denote

$$\alpha = \frac{2(p+1)}{p\theta-1}, \quad \beta = \frac{2(\theta+1)}{p\theta-1},$$

the scaling exponents of system (1.1).



**Proposition 3.1.** *Assume that  $N \geq 2 + 2x_0$  and  $\theta \geq p > 1$ . Let  $(u^*, v^*)$  be an extremal solution associated to (1.1). There exists  $\varepsilon_0 > 0$  such that if for some  $B_{R_0}(x) \subset \Omega$  with  $R_0 > 0$  and*

$$R_0^{2\beta-N} \int_{B_{R_0}(x)} (v^* + 1)^2 \leq \varepsilon_0,$$

*then  $x$  is a regular point of  $(u^*, v^*)$ , i.e.  $u^*, v^*$  are smooth in a neighborhood of  $x$ .*

For the proof of Proposition 3.1, we need to establish the following lemma.

**Lemma 3.5.** *There exist  $\varepsilon_1$  and  $\tau \in (0, 1)$  depending on  $N, p, \theta$  such that if  $(u; v)$  is a stable solution of (1.1),  $B_{R_0}(z) \subset \Omega$  and*

$$G_0 := R_0^{2\beta-N} \int_{B_{R_0}(z)} (v + 1)^2 dx \leq \varepsilon_1, \quad (3.1)$$

*then*

$$(\tau R_0)^{2\beta-N} \int_{B_{\tau R_0}(z)} (v + 1)^2 dx \leq \frac{1}{2} G_0. \quad (3.2)$$

**Proof.** By shifting coordinates, we can assume that  $z = 0$ . Up to the scaling

$$\tilde{u}(x) + 1 = R_0^\alpha (u(R_0 x) + 1), \quad \tilde{v}(x) + 1 = R_0^\beta (v(R_0 x) + 1), \quad (3.3)$$

we can assume  $R_0 = 1$  without loss of generality. By Lemma 3.1, we have  $2\theta \frac{p+1}{\theta+1} < s_0$ , hence by Lemma 3.4 and (1.8), there exist  $\ell \in \mathbb{N}$  and  $C > 0$  such that

$$\int_{B_{2^{-\ell}}} (u + 1)^{2\theta} \leq C \int_{B_{2^{-\ell}}} (v + 1)^{2\theta \frac{p+1}{\theta+1}} \leq C \left[ \int_{B_1} (v + 1)^2 \right]^{\frac{\theta(p+1)}{\theta+1}}.$$

Denote  $r_0 := 2^{-\ell}$  and using (3.1), we deduce that

$$\int_{B_{r_0}} (u + 1)^{2\theta} \leq C G_0^{\frac{\theta(p+1)}{\theta+1}} \leq C \varepsilon_1^{\frac{\theta(p+1)}{\theta+1}}. \quad (3.4)$$

Consider now the decomposition  $v + 1 = v_1 + v_2$  where

$$\begin{cases} -\Delta v_1 = 0 & \text{in } B_{r_0} \\ v_1 = v + 1 & \text{on } \partial B_{r_0}, \end{cases} \quad \begin{cases} -\Delta v_2 = \gamma(u + 1)^\theta & \text{in } B_{r_0} \\ v_2 = 0 & \text{on } \partial B_{r_0}. \end{cases}$$

Let  $0 < \tau < r_0$  (to be fixed later on), we have

$$\int_{B_\tau} (v + 1)^2 dx \leq 2 \int_{B_\tau} v_1^2 dx + 2 \int_{B_\tau} v_2^2 dx. \quad (3.5)$$

Noting that  $v_1^2$  is subharmonic in  $B_{r_0}$ , we get

$$\tau^{2\beta-N} \int_{B_\tau} v_1^2 dx \leq C \tau^{2\beta} \int_{B_{r_0}} v_1^2 dx \leq C \tau^{2\beta} \int_{B_1} (v + 1)^2 dx = C \tau^{2\beta} G_0. \quad (3.6)$$

On the other hand, by elliptic theory and (3.4), there holds, as  $G_0 \leq \varepsilon_1$ ,

$$\int_{B_{r_0}} v_2^2 \leq \|v_2\|_{H^2(B_{r_0})}^2 \leq C G_0^{\frac{\theta(p+1)}{\theta+1}} \leq C \varepsilon_1^{\frac{p\theta-1}{\theta+1}} G_0. \quad (3.7)$$

Combining (3.5)-(3.7), we obtain

$$\tau^{2\beta-N} \int_{B_\tau} (v+1)^2 dx \leq C\tau^{2\beta} G_0 + C\tau^{2\beta-N} \varepsilon_1^{\frac{p\theta-1}{\theta+1}} G_0.$$

Fix  $\tau > 0$  so that  $C\tau^{2\beta} \leq \frac{1}{4}$ . Then, take  $\varepsilon_1 > 0$  sufficiently small so that  $C\tau^{2\beta-N} \varepsilon_1^{\frac{p\theta-1}{\theta+1}} \leq \frac{1}{4}$ , we are done.  $\square$

**Proof of Proposition 3.1.** By approximating the extremal solution  $(u^*, v^*)$  of (1.1) by minimal solutions with parameters  $(\lambda, \gamma) \in \Gamma_\sigma$ , Lemma 3.5 holds true for  $v^*$ . As above, we can assume that  $x = 0$  and  $R_0 = 1$ .

Since  $N \geq 2 + 2x_0 = 2 + \beta s_0$  and  $s_0 > 2$ , we get  $N - 2\beta > 0$ . Let  $\varepsilon_1$  be the constant in Lemma 3.5 and choose  $\varepsilon_0$  such that  $2^{N-2\beta} \varepsilon_0 = \varepsilon_1$ . For any  $y \in B_{\frac{1}{2}}$ , we have

$$G_1 := \left(\frac{1}{2}\right)^{2\beta-N} \int_{B_{\frac{1}{2}}(y)} (v^* + 1)^2 dx \leq 2^{N-2\beta} \int_{B_1} (v^* + 1)^2 dx \leq 2^{N-2\beta} \varepsilon_0 = \varepsilon_1.$$

Applying inductively Lemma 3.5, then for any  $k \geq 1$ ,

$$\left(\frac{\tau^k}{2}\right)^{2\beta-N} \int_{B_{\frac{\tau^k}{2}}(y)} (v^* + 1)^2 dx \leq 2^{-k} G_1. \quad (3.8)$$

Let  $0 < \rho \leq \tau/2$ , we can take  $k \in \mathbb{N}^*$  such that  $\frac{\tau^{k+1}}{2} < \rho \leq \frac{\tau^k}{2}$ . Therefore,

$$\begin{aligned} \rho^{2\beta-N} \int_{B_\rho(y)} (v^* + 1)^2 dx &\leq \left(\frac{\tau^{k+1}}{2}\right)^{2\beta-N} \int_{B_{\frac{\tau^k}{2}}(y)} (v^* + 1)^2 dx \\ &\leq 2^{N-2\beta} \tau^{2\beta-N} 2^{-k} G_1 \\ &\leq C(N, \beta, \tau, \varepsilon_1) 2^{-k-1} \\ &\leq C\tau^{(k+1)\delta}, \end{aligned}$$

where  $\delta = \frac{-\ln 2}{\ln \tau} > 0$ . This implies that

$$\int_{B_\rho(y)} (v^* + 1)^2 dx \leq C\rho^{N-2\beta+\delta}, \quad \forall y \in B_{\frac{1}{2}}, \quad 0 < \rho \leq \frac{\tau}{2}. \quad (3.9)$$

Furthermore, applying Lemma 3.4 with  $s = p + 1$  and using (3.9), we get

$$\int_{B_\rho(y)} (v^* + 1)^{p+1} dx \leq C\rho^{N-(p+1)\beta+\delta}, \quad \forall y \in B_{\frac{1}{2}}, \quad 0 < \rho \leq \frac{\tau}{2^{\ell+1}}$$

for some integer  $\ell \geq 1$ . By approximation argument, the estimate (1.8) holds a.e. in  $\Omega$ , if we replace  $(u, v)$  by  $(u^*, v^*)$ . Therefore,

$$\int_{B_\rho(y)} (u^* + 1)^{\theta+1} dx \leq C\rho^{N-(p+1)\beta+\delta}, \quad \forall y \in B_{\frac{1}{2}}, \quad 0 < \rho \leq \frac{\tau}{2^{\ell+1}}.$$

This means that  $u^* + 1$  belongs to the Morrey space

$$L^{\theta+1, N-(p+1)\beta+\delta}(B_{\frac{1}{2}}) \subset L^{\theta, N-\theta\alpha+\frac{\theta}{\theta+1}\delta}(B_{\frac{1}{2}}).$$

Finally, applying Theorem 3.4 in [1], we get the claim.  $\square$

### 3.3. Proof of Theorem 1.2 completed

Let  $p + 1 < s < \frac{N}{N-2}s_0$  such that  $L(s) < 0$ . Let  $z \in \Omega$  verifying

$$\lim_{R \rightarrow 0} R^{\beta s - N} \int_{B_R(z)} (v^* + 1)^s dx = 0.$$

By Hölder's inequality, there holds

$$\lim_{R \rightarrow 0} R^{2\beta - N} \int_{B_R(z)} (v^* + 1)^2 dx = 0.$$

Applying Proposition 3.1,  $z$  is a regular point for  $(u^*, v^*)$ . This implies that the singular set

$$\mathcal{S} \subset \left\{ x \in \Omega : \limsup_{R \rightarrow 0} R^{\beta s - N} \int_{B_R(x)} (v^* + 1)^s dx > 0 \right\}.$$

Take first  $s_1 > p + 1$  such that  $L(s_1) < 0$ . Using (2.2), for minimal solution  $(u, v)$  of (1.1) with  $(\lambda, \gamma) \in \Gamma_\sigma$ , as  $\frac{p+2s_1-1}{2} > 2$ ,

$$\int_{\Omega} (v + 1)^2 \leq \int_{\Omega} (u + 1)^{\frac{\theta-1}{2}} (v + 1)^{\frac{p+2s_1-1}{2}} \leq C_{s_1}.$$

Passing to the limit, the above estimate yields that  $v^* + 1 \in L^2(\Omega)$ . By Lemma 3.4,  $(v^* + 1)^s \in L^1_{loc}(\Omega)$ , it follows that  $\mathcal{H}^{N-\beta s}(\mathcal{S}) = 0$  whenever  $N - \beta s > 0$ , see Theorem 5.3.4 in [14]. Tending  $s$  to  $\frac{N}{N-2}s_0$ , we conclude that the Hausdorff dimension of  $\mathcal{S}$  is less or equal than  $\max(N - \frac{2N}{N-2}s_0, 0)$ , recalling that  $x_0 = \frac{\theta+1}{p\theta-1}s_0$ . As  $N \geq 2 + 2x_0$ , the claim follows.  $\square$

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